Indian Statistical Institute
Midterm Examination 2015-2016
B.Math Third Year
Complex Analysis

Time : 3 Hours   Date : 08.09.2015   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $U \subseteq \mathbb{C}$ open. (v) $\mathrm{Hol}(U) = \{f : U \to \mathbb{C} \text{ holonorphic }\}$.

Q1. (10+10 = 20 marks) Evaluate the following integrations:

$$(i) \int_{C_1(0)} \frac{z}{2z+1}\, dz, \qquad (ii) \int_{C_1(0)} \frac{\sin z}{z^{38}}\, dz.$$

Q2. (15 marks) Find the maximum of $|e^{z^2}|$ on $\overline{B_1(0)}$.

Q3. (15 marks) Let f be a non-constant entire function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Q4. (15 marks) Let $f \in \mathrm{Hol}(\mathbb{C})$ and

$$|f(z)| \leq a(1 + |z|)^n, \qquad \forall z \in \mathbb{C},$$

for some positive real number a and natural number n. Prove that f is a polynomial of degree at most n.

Q5. (15 marks) Let $f \in \mathrm{Hol}(U)$ and f' is continuous on U. Use Green's theorem to show that

$$\int_{\partial R} f dz = 0,$$

for all closed and solid rectangle $R \subseteq U$.

Q6. (15 marks) Let f and g be two entire and linearly independent functions (that is, $f, g \in \mathrm{Hol}(\mathbb{C})$ and $f \neq \alpha g$ for any $\alpha \in \mathbb{C}$). Prove that there exists a sequence $\{z_n\} \subseteq \mathbb{C}$ such that

$$|f(z_n)| \geq |g(z_n)|, \qquad \forall n.$$

Q7. (15 marks) Let $f \in \mathrm{Hol}(B_{1+\epsilon}(0))$ for some $\epsilon > 0$ and $f(C_1(0)) \subseteq \mathbb{R}$. Prove that f is a constant function.